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Further Studies in Aesthetic Field Theory VII

M. MURASKIN
Physics Department, University of North Dakota, Grand Forks, North Dakota

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Abstract

We have generalized the field equations in aesthetic field theory by treating subscripted indices in a different manner than superscripted indices when considering the change of tensor functions. In doing so we find that scalar fields appear in the theory. We also have an infinite number of integrability equations that have to be satisfied. We have found a set of data at the origin such that these integrability equations are satisfied to computer accuracy, implying local existence of solutions. However, we have no evidence suggesting global existence at present.

1. *Introduction*

We have been studying the consequences of various aesthetic mathematical ideas for some time.

At this point we can claim the following results:

(1) It is possible to formulate a field theory based on mathematically aesthetic ideas.
(2) Solutions to the resulting field equations have been proved to exist locally (Muraskin, 1972).
(3) The field equations are capable of leading to sinusoidal behavior along a coordinate axis (Muraskin, 1974b).
(4) There is no sign, in some of our computer studies, of any singularities developing (Muraskin, 1973a, 1973b; Muraskin & Ring, 1973, 1974b). This suggests that solutions may exist globally as well.
(5) We have obtained bounded particle-like systems (Muraskin, 1973a, 1973b, 1974; Muraskin & Ring, 1973, 1974a, 1974b).
(6) Our particle-like object can be looked at as a two-particle system since we have a maximum center in proximity to a minimum center. So far as we know, no other author has obtained a two-particle system from a non-linear equation. We have been able to study trajectories of the two particles in the vicinity of one another.

Other authors have studied particle-like behavior in non-linear field equations (see some of our previous papers for reference). They do not attempt to motivate their equations according to aesthetic principles. Nor have they been able to obtain all the results we have described above.

However, there are some features of our work which do not appear to be satisfactory.

(1) The trajectories of the two particles appear to be straight lines even in the vicinity of one another.

(2) Outside the particle system we got little structure to the vacuum.

(3) There is no indication that our particle can be considered stable. The magnitude of the field associated with particle seems to get less and less in time.

Thus, the problem is whether we can improve upon our past results. From our more recent papers, Muraskin (1974) and Muraskin & Ring (1974b), we see that this is not a simple job.

2. *Generalized Field Equations*

In our original work, Muraskin (1970), we saw that all scalar fields constructed from $A_i, A^i, g_{ij}, g^{jk}, \Gamma^i_{jk}, \partial_i \Gamma^m_{jk}$, etc. were constant. We may ask whether it is possible to introduce such scalar fields in a natural way and still preserve our basic approach. We find in this paper that it is possible to do so.

For the change of a vector field A_i, we write as previously

$$dA_i = \Gamma^j_{ik} A_j \, dx^k \tag{2.1}$$

In the case that all scalars are constants we have

$$dA^i = -\Gamma^i_{jk} A^j \, dx^k \tag{2.2}$$

In order to obtain a greater generalization we may suppose that the change of A^i should be described by an entirely new function. Thus, we write

$$dA^i = -\theta^i_{jk} A^j \, dx^k \tag{2.3}$$

It then follows that

$$d(A_i A^i) = A_j A^i (\Gamma^j_{ik} - \theta^j_{ik}) \, dx^k \tag{2.4}$$

These scalar fields are no longer constant in such a theory. As in our previous work we can assume that Γ^i_{jk} acts like a product of three vectors of the type $A^i B_j C_k$ so far as its change is concerned.† Similarly θ^i_{jk} also is to act like a product of three vectors. Thus we get for the field equations

† We will not assume that Γ^i_{jk} behaves like $A^i B_j C_k E_m D^m$ due to the assumed fundamental character of Γ^i_{jk}.

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} = -\Gamma^m_{jk}\theta^i_{ml} + \Gamma^i_{mk}\Gamma^m_{jl} + \Gamma^i_{jm}\Gamma^m_{kl}$$

$$\frac{\partial \theta^i_{jk}}{\partial x^l} = -\theta^m_{jk}\theta^i_{ml} + \theta^i_{mk}\Gamma^m_{jl} + \theta^i_{jm}\Gamma^m_{kl} \tag{2.5}$$

If $\theta^i_{jk} = \Gamma^i_{jk}$, the equations collapse into the field equations $\Gamma^i_{jk;l} = 0$ considered in our earlier work.

The equations for g_{ij} and g^{ij} read

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^t_{ik}g_{tj} + \Gamma^t_{jk}g_{it}$$

$$\frac{\partial g^{ij}}{\partial x^k} = -\theta^i_{tk}g^{tj} - \theta^j_{tk}g^{it} \tag{2.6}$$

We note that g_{ij} and g^{jk} in (2.6) do not obey $g_{ij}g^{jk} = \delta_i^{\ k}$ at all points so long as $\theta^i_{jk} \neq \Gamma^i_{jk}$. Thus, tensors having contracted indices will not obey the same change equations as tensors not having contracted indices. We note that we cannot have $g_{ij}g^{jk} = \delta_i^{\ k}$ at all points if g_{ij} and g^{ij} as well as Γ^i_{jk} and θ^i_{jk} go to zero at infinity.

We may ask if our field equations are consistent with integrability. From

$$\frac{\partial^2 A_i}{\partial x^j\,\partial x^k} = \frac{\partial^2 A_i}{\partial x^k\,\partial x^j} \tag{2.7}$$

we get

$$\Gamma^j_{it}\Gamma^t_{kl} - \Gamma^j_{it}\Gamma^t_{lk} + \Gamma^t_{il}\theta^j_{tk} - \Gamma^t_{ik}\theta^j_{tl} = 0 \tag{2.8}$$

From

$$\frac{\partial^2 A^i}{\partial x^j\,\partial x^k} = \frac{\partial^2 A^i}{\partial x^k\,\partial x^j} \tag{2.9}$$

we get

$$\theta^i_{jm}\Gamma^m_{lk} - \theta^i_{jm}\Gamma^m_{kl} + \Gamma^m_{jk}\theta^i_{ml} - \Gamma^m_{jl}\theta^i_{mk} = 0 \tag{2.10}$$

(2.8) and (2.10) together imply

$$\Gamma^i_{jk}(\Gamma^m_{kl} - \Gamma^m_{lk}) = \theta^i_{jm}(\Gamma^m_{kl} - \Gamma^m_{lk}) \tag{2.11}$$

This equation is satisfied if we take $\Gamma^i_{jk} = \theta^i_{jk}$ or if $\Gamma^i_{jk} = \Gamma^i_{kj}$. We shall consider $\Gamma^i_{jk} = \Gamma^i_{kj}$ from now on. Then (2.8) becomes

$$\Gamma^t_{il}\theta^j_{tk} - \Gamma^t_{ik}\theta^j_{tl} = 0 \tag{2.12}$$

This equation becomes the basic integrability equation for our field theory at the origin. We see this, as follows. From

$$\frac{\partial^2 g_{ij}}{\partial x^l\,\partial x^k} = \frac{\partial^2 g_{ij}}{\partial x^k\,\partial x^l} \tag{2.13}$$

we get

$$g_{tj}(\Gamma^m_{il}\theta^t_{mk} - \Gamma^m_{ik}\theta^t_{ml}) + g_{it}(\Gamma^m_{jl}\theta^t_{mk} - \Gamma^m_{jk}\theta^t_{ml}) = 0 \tag{2.14}$$

From

$$\frac{\partial^2 g^{ij}}{\partial x^m\, \partial x^k} = \frac{\partial^2 g^{ij}}{\partial x^k\, \partial x^m} \tag{2.15}$$

we get

$$g^{it}(\Gamma^l_{tk}\theta^j_{lm} - \Gamma^l_{tm}\theta^j_{lk}) + g^{tj}(\Gamma^l_{tk}\theta^i_{lk} - \Gamma^l_{tm}\theta^i_{lk}) = 0 \tag{2.16}$$

From

$$\frac{\partial^2 \Gamma^i_{jk}}{\partial x^t\, \partial x^l} = \frac{\partial^2 \Gamma^i_{jk}}{\partial x^l\, \partial x^t} \tag{2.17}$$

we get

$$\Gamma^i_{mk}(\Gamma^p_{jt}\theta^m_{pl} - \Gamma^p_{jl}\theta^m_{pt}) + \Gamma^i_{jm}(\Gamma^p_{kt}\theta^m_{pl} - \Gamma^p_{kl}\theta^m_{pt}) - \Gamma^m_{jk}(\Gamma^p_{mt}\theta^i_{pl} - \Gamma^p_{ml}\theta^i_{pt}) = 0 \tag{2.18}$$

From

$$\frac{\partial^2 \theta^i_{jk}}{\partial x^t\, \partial x^l} = \frac{\partial^2 \theta^i_{jk}}{\partial x^l\, \partial x^t} \tag{2.19}$$

we get

$$\theta^i_{mk}(\Gamma^p_{jt}\theta^m_{pl} - \Gamma^p_{jl}\theta^m_{pt}) + \theta^i_{jm}(\Gamma^p_{kt}\theta^m_{pl} - \Gamma^p_{kl}\theta^m_{pt}) - \theta^m_{jk}(\Gamma^p_{mt}\theta^i_{pl} - \Gamma^p_{ml}\theta^i_{pt}) = 0 \tag{2.20}$$

All these restrictions, (2.14), (2.16), (2.18) and (20.2), are satisfied identically once (2.12) is satisfied. Also

$$\frac{\partial^2 (A_i B^i)}{\partial x^k\, \partial x^l} = \frac{\partial^2 (A_i B^i)}{\partial x^l\, \partial x^k} \tag{2.21}$$

is satisfied so long as Γ^i_{jk} is symmetric in the bottom two indices. We then can see that the mixed derivatives of all functions of $A_i, A^i, g_{ij}\, g^{ij}, \Gamma^i_{jk}, \theta^i_{jk}$ and ∂_i are symmetric provided that (2.12) is satisfied. We use the fact that ∂_i acting on any function of $A_i, A^i, g_{ij}, g^{ij}, \Gamma^i_{jk}, \theta^i_{jk}$ can be expressed using the change equations (2.1), (2.2), (2.5) and (2.6) in terms of products of these variables.

However, if the integrability equations are satisfied at one point this does not mean that they are satisfied at all points. We consider the derivative of the function appearing in (2.12)

$$\frac{\partial}{\partial x^l}(\Gamma^t_{im}\theta^j_{tk} - \Gamma^t_{ik}\theta^j_{tm}) \tag{2.22}$$

Using the fact that (2.12) is satisfied at the origin we get that this quantity is not zero in general, but is equal to

$$(\Gamma^t_{pl} - \theta^t_{pl})(\Gamma^p_{im}\theta^j_{tk} - \Gamma^p_{ik}\theta^j_{tm}) \tag{2.23}$$

Thus, as in Muraskin (1973a), it is necessary that an infinite number of integrability equations are needed for a consistent theory. All derivatives of (2.12) should be zero so that the integrability conditions are satisfied at all points. We shall show that these conditions can be satisfied to computer accuracy in Section 3 by using an appropriate set of data at the origin.

In looking for a maximum or minimum for g_{00} we find that the equation for $e^0{}_k$ ($k = 1, 2, 3$) is the same as Muraskin (1971). However, the formula for A_{jk} should be replaced by

$$\begin{aligned} A_{jk} = 2(&g_{t0}\Gamma^t_{mk}\Gamma^m_{0j} + g_{t0}\Gamma^t_{0m}\Gamma^m_{kj} - g_{t0}\Gamma^m_{0k}\theta^t_{mj} \\ &+ g_{t0}\Gamma^m_{0k}\Gamma^t_{mj} + g_{tm}\Gamma^t_{0k}\Gamma^m_{0j}) \end{aligned} \tag{2.24}$$

Once we have a simple set of $g_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma}$, $\theta^\alpha_{\beta\gamma}$ satisfying the integrability equations at a point we can get a more complicated set by using

$$g_{ij} = e^\alpha{}_i e^\beta{}_j g_{\alpha\beta} \tag{2.25}$$

$$\Gamma^i_{jk} = e_\alpha{}^i e^\beta{}_j e^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \tag{2.26}$$

$$\theta^i_{jk} = e_\alpha{}^i e^\beta{}_j e^\gamma{}_k \theta^\alpha_{\beta\gamma} \tag{2.27}$$

$$e^\alpha{}_i e_\alpha{}^j = \delta_i{}^j \tag{2.28}$$

3. *Solution of Integrability Equations and Computer Work*

We consider the following data:

$\Gamma^\alpha_{\beta\gamma}$:

$\Gamma^1_{11} = -0{\cdot}1$	$\Gamma^1_{12} = 0{\cdot}1$	$\Gamma^1_{13} = -0{\cdot}1$	$\Gamma^1_{10} = 0$
$\Gamma^1_{21} = 0{\cdot}1$	$\Gamma^1_{22} = -0{\cdot}1$	$\Gamma^1_{23} = -0{\cdot}1$	$\Gamma^1_{20} = 0$
$\Gamma^1_{31} = -0{\cdot}1$	$\Gamma^1_{32} = -0{\cdot}1$	$\Gamma^1_{33} = 0{\cdot}1$	$\Gamma^1_{30} = 0$
$\Gamma^1_{01} = 0$	$\Gamma^1_{02} = 0$	$\Gamma^1_{03} = 0$	$\Gamma^1_{00} = 0$
$\Gamma^2_{11} = -0{\cdot}1$	$\Gamma^2_{12} = -0{\cdot}1$	$\Gamma^2_{13} = 0{\cdot}1$	$\Gamma^2_{10} = 0$
$\Gamma^2_{21} = -0{\cdot}1$	$\Gamma^2_{22} = 0{\cdot}1$	$\Gamma^2_{23} = -0{\cdot}1$	$\Gamma^2_{20} = 0$
$\Gamma^2_{31} = 0{\cdot}1$	$\Gamma^2_{32} = -0{\cdot}1$	$\Gamma^2_{33} = -0{\cdot}1$	$\Gamma^2_{30} = 0$
$\Gamma^2_{01} = 0$	$\Gamma^2_{02} = 0$	$\Gamma^2_{03} = 0$	$\Gamma^2_{00} = 0$
$\Gamma^3_{11} = 0{\cdot}1$	$\Gamma^3_{12} = -0{\cdot}1$	$\Gamma^3_{13} = -0{\cdot}1$	$\Gamma^3_{10} = 0$
$\Gamma^3_{21} = -0{\cdot}1$	$\Gamma^3_{22} = -0{\cdot}1$	$\Gamma^3_{23} = 0{\cdot}1$	$\Gamma^3_{20} = 0$
$\Gamma^3_{31} = -0{\cdot}1$	$\Gamma^3_{32} = 0{\cdot}1$	$\Gamma^3_{33} = -0{\cdot}1$	$\Gamma^3_{30} = 0$
$\Gamma^3_{01} = 0$	$\Gamma^3_{02} = 0$	$\Gamma^3_{03} = 0$	$\Gamma^3_{00} = 0$
$\Gamma^0_{11} = 0$	$\Gamma^0_{12} = 0$	$\Gamma^0_{13} = 0$	$\Gamma^0_{10} = 0$
$\Gamma^0_{21} = 0$	$\Gamma^0_{22} = 0$	$\Gamma^0_{23} = 0$	$\Gamma^0_{20} = 0$
$\Gamma^0_{31} = 0$	$\Gamma^0_{32} = 0$	$\Gamma^0_{33} = 0$	$\Gamma^0_{30} = 0$
$\Gamma^0_{01} = 0$	$\Gamma^0_{02} = 0$	$\Gamma^0_{03} = 0$	$\Gamma^0_{00} = 0{\cdot}1$ (3.1a)

$\theta^{\alpha}_{\beta\gamma}$:

$$
\begin{array}{llll}
\theta^1_{11} = 0{\cdot}1 & \theta^1_{12} = 0{\cdot}1 & \theta^1_{13} = -0{\cdot}1 & \theta^1_{10} = 0 \\
\theta^1_{21} = 0{\cdot}1 & \theta^1_{22} = -0{\cdot}1 & \theta^1_{23} = 0{\cdot}1 & \theta^1_{20} = 0 \\
\theta^1_{31} = -0{\cdot}1 & \theta^1_{32} = 0{\cdot}1 & \theta^1_{33} = 0{\cdot}1 & \theta^1_{30} = 0 \\
\theta^1_{01} = 0 & \theta^1_{02} = 0 & \theta^1_{03} = 0 & \theta^1_{00} = 0 \\
\theta^2_{11} = -0{\cdot}1 & \theta^2_{12} = 0{\cdot}1 & \theta^2_{13} = 0{\cdot}1 & \theta^2_{10} = 0 \\
\theta^2_{21} = 0{\cdot}1 & \theta^2_{22} = 0{\cdot}1 & \theta^2_{23} = -0{\cdot}1 & \theta^2_{20} = 0 \\
\theta^2_{31} = 0{\cdot}1 & \theta^2_{32} = -0{\cdot}1 & \theta^2_{33} = 0{\cdot}1 & \theta^2_{30} = 0 \\
\theta^2_{01} = 0 & \theta^2_{02} = 0 & \theta^2_{03} = 0 & \theta^2_{00} = 0 \\
\theta^3_{11} = 0{\cdot}1 & \theta^3_{12} = -0{\cdot}1 & \theta^3_{13} = 0{\cdot}1 & \theta^3_{10} = 0 \\
\theta^3_{21} = -0{\cdot}1 & \theta^3_{22} = 0{\cdot}1 & \theta^3_{23} = 0{\cdot}1 & \theta^3_{20} = 0 \\
\theta^3_{31} = 0{\cdot}1 & \theta^3_{32} = 0{\cdot}1 & \theta^3_{33} = -0{\cdot}1 & \theta^3_{30} = 0 \\
\theta^3_{01} = 0 & \theta^3_{02} = 0 & \theta^3_{03} = 0 & \theta^3_{00} = 0 \\
\theta^0_{11} = 0 & \theta^0_{12} = 0 & \theta^0_{13} = 0 & \theta^0_{10} = 0 \\
\theta^0_{21} = 0 & \theta^0_{22} = 0 & \theta^0_{23} = 0 & \theta^0_{20} = 0 \\
\theta^0_{31} = 0 & \theta^0_{32} = 0 & \theta^0_{33} = 0 & \theta^0_{30} = 0 \\
\theta^0_{01} = 0 & \theta^0_{02} = 0 & \theta^0_{03} = 0 & \theta^0_{00} = 0{\cdot}1
\end{array} \tag{3.1b}
$$

$\Gamma^{\alpha}_{\beta\gamma}$ and $\theta^{\alpha}_{\beta\gamma}$ are both symmetric in the bottom two indices. $g_{\alpha\beta}$ was taken to be diag(1, 1, 1, 1). We have found that (2.12) and (2.23) are zero both at the origin and away from the origin, to computer accuracy. Also, we ran down the x-axis followed by the y-axis, together with the case when the order was reversed. We found that the answers for the field values agreed to computer accuracy.

Thus it appears that our theory based on Γ^i_{jk}, θ^i_{jk} is consistent, at least in the vicinity of the origin.

We chose $e^{\alpha}{}_i$ to be

$$
\begin{array}{llll}
e^1{}_1 = 0{\cdot}24 & e^1{}_2 = 0{\cdot}29 & e^1{}_3 = 0.35 & e^1{}_0 = 1{\cdot}2 \\
e^2{}_1 = -0{\cdot}02 & e^2{}_2 = 0{\cdot}9 & e^2{}_3 = -0{\cdot}03 & e^2{}_0 = 0{\cdot}082 \\
e^3{}_1 = -0{\cdot}015 & e^3{}_2 = -0{\cdot}017 & e^3{}_3 = 0{\cdot}85 & e^3{}_0 = 0{\cdot}092 \\
 & & & e^0{}_0 = 2{\cdot}0
\end{array} \tag{3.2}
$$

$e^0{}_1, e^0{}_2, e^0{}_3$ were chosen to make g_{00} an extremum at the origin. We have not, as yet, found a situation in which g_{00} is a maximum or minimum at the origin. Running down the x-axis we have obtained the following results for g_{00}.

x	g_{00}
0	5·46
1	5·46
2	5·47
4·3	5·52

10	5·77
15·25	6·15
35·25	8·87
65·25	21·75
100·25	243·82
110·25	2357

Thus the values for g_{00} are getting increasingly bigger. Either we are dealing with large numbers, or else a singularity could be developing.

We tried a second set of $e^{\alpha}{}_{i}$,

$$\begin{array}{llll} e^{1}{}_{1} = 0{\cdot}88 & e^{1}{}_{2} = -0{\cdot}42 & e^{1}{}_{3} = -0{\cdot}32 & e^{1}{}_{0} = 0{\cdot}19 \\ e^{2}{}_{1} = 0{\cdot}5 & e^{2}{}_{2} = 0{\cdot}9 & e^{2}{}_{3} = -0{\cdot}425 & e^{2}{}_{0} = 0{\cdot}3 \\ e^{3}{}_{1} = 0{\cdot}2 & e^{3}{}_{2} = -0{\cdot}55 & e^{3}{}_{3} = 0{\cdot}89 & e^{3}{}_{0} = 0{\cdot}6 \\ e^{0}{}_{1} = 0{\cdot}44 & e^{0}{}_{2} = -0{\cdot}16 & e^{0}{}_{3} = 0{\cdot}39 & e^{0}{}_{0} = 1{\cdot}01 \end{array} \tag{3.3}$$

The results for the x-axis run are as follows:

x	g_{00}
0	1·51
1	1·51
2	1·56
4	1·77
6	2·11
12	4·743
18	23·70
22	996·3
22·71	$0{\cdot}18 \times 10^{7}$

We have also compared the results in this last run with Γ^{i}_{jk} associated with $\Gamma^{i}_{jk;l} = 0$ and having $\Gamma^{\alpha}_{\beta\gamma}$ given by (3.1a) and $e^{\alpha}{}_{i}$ by (3.3). The values of Γ^{i}_{jk} are thus the same for the two cases at the origin. Away from the origin the values of Γ^{i}_{jk} differ. However, at $x = 22{\cdot}71$ we found little difference in Γ^{i}_{jk} for the two cases. At this point fifty-six of the sixty-four Γ^{i}_{jk} were approaching each other when comparison was made for the two cases. The other eight did not fit into this pattern as well. The rate of difference was increasing for seven out of these. On the other hand, the largest difference in this latter seven was only 0·0014 for Γ^{1}_{30}. The value of Γ^{1}_{30} at this point was $-66{\cdot}5$. The difference for all Γ^{i}_{jk} was less than 0·01 in magnitude. The gammas ranged form 2·2 to 270 at this point. The conclusion we reach is that it appears thus far that we have not found any significant new effects associated with the introduction of θ^{i}_{jk} for the set of origin data we have used.

4. *Conclusion*

It is a natural extension of our previous work to treat subscripted indices different from superscripted indices when considering the change of tensor functions. In doing so, scalar functions of the basic fields appear in the theory.

We have been able to demonstrate to computer accuracy that the infinite number of integrability equations associated with our generalized field equations are satisfied. This indicates that solutions to the field equations exist locally.

This would indicate that the new theory has some promise. However, we have run into the following difficulties.

(1) It is not a simple matter to find other solutions to the infinite number of integrability equations.
(2) It has been difficult to obtain a maximum or minimum in g_{00} at the origin.
(3) The values for g_{00} have not been shown to be bounded when we run down an axis.
(4) The presence of $\Gamma^i_{jk} \neq \theta^i_{jk}$ does not appear to significantly alter the results obtained when $\Gamma^i_{jk} = \theta^i_{jk}$ for the data we have been using.

Also, we have not been able to combine $O(3)$ invariant data with the present system of equations.

It remains to be seen whether the above difficulties can be overcome.

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References

Muraskin, M. (1970). *Annals of Physics*, **59**, 27.
Muraskin, M. (1971). *International Journal of Theoretical Physics*, Vol. 4, No. 1, p. 49.
Muraskin, M. (1972). *Journal of Mathematical Physics*, **13**, 863.
Muraskin, M. (1973a). *International Journal of Theoretical Physics*, Vol. 7, No. 1, p. 213.
Muraskin, M. (1973b). *International Journal of Theoretical Physics*, Vol. 8, No. 2, p. 93.
Muraskin, M. (1974). To be published.
Muraskin, M. and Ring, Beatrice (1972). *International Journal of Theoretical Physics*, Vol. 6, No. 2, p. 105.
Muraskin, M. and Ring, Beatrice (1973a). *International Journal of Theoretical Physics*, Vol. 8, No. 2, p. 85.
Muraskin, M. and Ring, Beatrice (1974a). *Foundations of Physics*, **4**, 395.
Muraskin, M. and Ring, Beatrice (1974b). *International Journal of Theoretical Physics*, Vol. 11, No. 2, p. 93.